Crumpled phases of self-avoiding randomly polymerized membranes

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We analyze the crumpled phases of self-avoiding two-dimensional polymerized membranes (tethered membrane) in d dimensions with disorders, using a Gaussian variational approximation. We find that a stress disorder, even a short-range one, is relevant in the crumpled phase at $d<8$ and alters the behaviors of the membranes. We give the exponent for the radius of gyration in the crumpled phases and find that the self-avoiding tethered membranes with short-range disorders can be in the crumpled phase at $d>2$ in contrast to the pure self-avoiding tethered membrane. We also give the phase diagrams of the crumpled phases of the self-avoiding polymerized membranes with long-range disorders.

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I. INTRODUCTION

There has been considerable interest in polymerized (tethered) membranes with disorders. Mutz, Bensimon, and Brienne [1] observed a phase transition in partially polymerized membranes of diacetylenic phospholipids. These membranes undergo a spontaneous transition to the wrinkled and rigid structure upon cooling. This transition can be attributed to the spin-glass transition [1–3]. Nelson and Radzihovsky [4,5] and Morse, Lubensky, and Grest [6] considered the phantom tethered membranes with randomness by the field theoretical method ($\epsilon=4-D$ expansion). It was concluded that at $T>0$ weak short-range disorders have only negligible effect at large scales and that the membrane remains in the pure flat phase. This problem has been studied also in the large d limit by Radzihovsky and Le Dousaal [7]. They showed that for a large enough disorder strength the flat phase at $T>0$ becomes unstable towards a crumpled-glass phase. Another possibility is that a more drastic type of disorder is needed to destabilize the flat phase. In their work [5], Nelson and Radzihovsky pointed out that unscreened disclinations would generate random stresses with long-range correlations and destabilize the flat phase at all temperatures. Using a self-consistent screening approximation, Le Doussal and Radzihovsky studied the flat phases of randomly polymerized membranes with long-range disorders [8]. Numerical simulation has been carried out [9]. However, the self-avoidance cannot be neglected in the physics of tethered membranes. For example, numerical simulations [10–19] showed that self-avoiding tethered membranes are always flat and several theoretical works were presented in order to explain this phenomenon [20–22]. From this view point, we considered randomly polymerized membranes with long-range interactions using the large d limit and discussed the behaviors of the transition [23]. Making use of its relation to the self-avoidance case, we presented a conjecture that the self-avoiding randomly polymerized membrane ($d=3$, $D=2$) is in the flat phase or in the crumpled-glass phase. However, we neglected the fluctuation of the spin-glass operator and, for the finite d case, this fluctuation becomes very important. In this paper, we incorporate this fluctuation at the one-loop level by using a Gaussian variational approximation and discuss the behaviors of the crumpled phases of the self-avoiding randomly polymerized membranes.

The paper consists of the following. In Sec. II we define our model and using the Gaussian variational method we obtain the saddle point equations. We discuss the saddle point equations with and without disorders. The difference was not considered in our previous work [23]. In Sec. III we obtain the large distance behaviors of the membrane for several types of randomness by analyzing the saddle point equations. We show that even a short-range stress disorder is relevant at $d<8$ and the crumpled phase exists at $2<d\leq 4$. This means that the self-avoiding polymerized membranes with short-range stress disorder can be crumpled at the physical dimension ($d=3$). We also obtain the phase diagrams of the crumpled phases of the self-avoiding polymerized membranes with long-range disorders. Section IV contains summary and concluding remarks. In the Appendix we discuss the relevance of the self-avoidance in the flat phases of tethered membranes in the Gaussian variational approach. We complete the connection between the long-range repulsive interaction case and the short-range self-avoiding case.

II. FREE ENERGY AND VARIATIONAL METHOD

We consider a D-dimensional membrane in a d-dimensional space. The position of the membrane is described by d bulk coordinates $X^i(\sigma^\alpha)$ ($i=1,\ldots,d$), where σ^α ($\alpha=1,\ldots,D$) are the internal manifold coordinates. We denote by u the strength of the "excluded volume" interaction. The Hamiltonian for the generalized Edwards model is given by [24]

$$\mathcal{H}[\mathbf{X}(\sigma)]=\int d^D\sigma\tfrac{1}{2}\partial_\alpha X^i\partial_\alpha X^i +u\int d^D\sigma\int d^D\sigma'\delta^d(X^i(\sigma)-X^i(\sigma')) \ . \quad (2.1)$$

The first term corresponds to the Gaussian potential of a free tethered manifold. In order to take into account the randomness of the stress and spontaneous curvature of the manifold, we introduce random fields $\delta c(\sigma)$ and $H^i(\sigma)$ [5] whose distributions are Gaussian. That is, in Fourier space we have

$$\langle \delta c(q_1)\delta c(q_2)\rangle_{\rm dis}=\Delta_\mu(q_1)\delta^D(q_1+q_2) =\Delta'_\mu q_1^{-z_\mu}\delta^D(q_1+q_2)\ , \qquad (2.2)$$

$$\langle H^i(q_1)H^j(q_2)\rangle_{\rm dis}=\delta_{ij}\Delta_K(q_1)\delta^D(q_1+q_2) =\delta_{ij}\Delta'_K q_1^{-z_K}\delta^D(q_1+q_2)\ . \qquad (2.3)$$

Here $\langle\ \rangle_{\rm dis}$ denotes the average over disorders.

Then, we start with the following Hamiltonian:

$$\mathcal{F}_{C,\mathbf{H}}[\mathbf{X}(\sigma)]=\int d^D\sigma \tfrac{1}{2}\partial_\alpha X^i\partial_\alpha X^i +\int d^D\sigma[\delta c(\sigma)\partial_\alpha X^i\partial_\alpha X^i+H^i(\sigma)\Delta X^i] +u\int d^D\sigma\int d^D\sigma'\delta^d(X^i(\sigma)-X^i(\sigma'))\ . \qquad (2.4)$$

The disorder-averaged effective free energy $\mathcal{F}_{\rm eff}$ is given by the average of the logarithm of the partition function $Z_{C,\mathbf{H}}$ for each configuration of the randomness. We have recourse to the replica formalism [25] and introduce n copies of the fields $\mathbf{X}$ labeled by the replica index a. The total Hamiltonian is the replicated version of the Hamiltonian (2.4),

$$\mathcal{F}_{\rm total}=\sum_{a=1}^{n}\mathcal{F}_{C,\mathbf{H}}[\mathbf{X}_a]\ . \qquad (2.5)$$

We take an average of the replicated partition function over the randomness to get the replicated Hamiltonian $\mathcal{F}_{\rm rep}$,

$$\langle e^{-\mathcal{F}_{\rm total}}\rangle_{\rm dis}=e^{-\mathcal{F}_{\rm rep}}\ , \qquad (2.6)$$

where

$$\mathcal{F}_{\rm rep}=\tfrac{1}{2}\int d^Dk\, K_{ab}(k)X_a^i(-k)X_b^i(k) -\tfrac{1}{2}\sum_{a,b=1}^{n}\int_{q=k_1+k_2,k_1+k_2=-(k_3+k_4)}d^Dk_1d^Dk_2d^Dk_3d^Dk_4\Delta_\mu(q)k_1^\alpha k_2^\alpha X_a^i(k_1)X_a^i(k_2)k_3^\beta k_4^\beta X_b^j(k_3)X_b^j(k_4) +\sum_{a=1}^{n}u\int d^D\sigma\int d^D\sigma'\delta^d(X_a^i(\sigma)-X_a^i(\sigma'))\ . \qquad (2.7)$$

In the above, $K_{ab}(k)=k^2\delta_{ab}-\Delta_K(k)k^4J_{ab}$, where $J_{ab}=1$ for all a,b. To calculate the effective free energy we use the Gaussian variational approximation [20,26,27]. The method consists in choosing as a variational Hamiltonian the most general quadratic form

$$\mathcal{H}_{\rm var}=\tfrac{1}{2}\int d^Dk\, X_a^i(-k)G_{ab}^{-1}(k)X_b^i(k)\ . \qquad (2.8)$$

That is, two-point correlation function is given by

$$\langle X_a^i(-k)X_b^j(k)\rangle_{\rm var}=\delta_{ij}G_{ab}(k)\ , \qquad (2.9)$$

where $\langle\ \rangle_{\rm var}$ means the thermal average with the trial Hamiltonian (2.8). Then the effective free energy is given by [26]

$$\frac{1}{L^D}\mathcal{F}_{\rm eff}=\lim_{n\to 0}\frac{1}{n}\left[\frac{1}{L^D}\langle\mathcal{F}_{\rm rep}-\mathcal{H}_{\rm var}\rangle_{\rm var}-\frac{d}{2}{\rm Tr}\ln G_{ab}(k)\right]\ , \qquad (2.10)$$

where L is the linear size of the membrane. It is easy to carry out this calculation to find

$$\frac{1}{L^D}\langle\mathcal{F}_{\rm rep}-\mathcal{H}_{\rm var}\rangle_{\rm var}+\frac{d}{2}{\rm Tr}\ln G_{ab}^{-1}(k)=\frac{d}{2}\int\frac{d^Dk}{(2\pi)^D}[K_{ab}(k)G_{ab}-1]-\frac{d}{2}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_3}{(2\pi)^D}[n^2\Delta_\mu(0)\mathbf{k}_1^2\mathbf{k}_3^2G(k_1)G_{bb}(k_3)] -\frac{d^2}{2}\int_{q=k_1+k_2}\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\Delta_\mu(q)\sum_{a,b}[2(\mathbf{k}_1\cdot\mathbf{k}_2)^2G_{ab}(k_1)G_{ab}(k_2)] -\frac{d}{2}\int\frac{d^Dk}{(2\pi)^D}{\rm Tr}\ln G_{ab}^{-1}(k)+\frac{u}{(2\pi)^d}\int d^D\sigma\left[\frac{\pi}{K(\sigma)}\right]^{d/2}\ , \qquad (2.11)$$

where $K(\sigma)$is the two-point correlation function for the trial Hamiltonian,

$$K(\sigma)=\lim_{n\to 0}\frac{1}{n}\sum_{a=1}^{n}\frac{1}{2d}\langle (X_a^i(\sigma)-X_a^i(0))^2\rangle_{\rm var}$$
$$=\lim_{n\to 0}\frac{1}{n}\sum_{a=1}^{n}\int\frac{d^Dk}{(2\pi)^D}[1-\cos(\mathbf{k}\cdot\boldsymbol{\sigma})]G_{aa}(k)\ . \qquad (2.12)$$

We now look for a replica symmetric solution

$$G_{ab}(k)=g_K(k)\delta_{ab}+g_\Delta(k)J_{ab}\ . \qquad (2.13)$$

For later convenience, we introduce the following expression for the "bare" propagators:

$$K_{ab}(k)=\delta_{ab}h_K(k)-J_{ab}h_\Delta(k)\ . \qquad (2.14)$$

With these propagators, the effective free energy is evaluated as

$$\frac{1}{L^D}\mathcal{F}_{\rm eff}=\frac{d}{2}\int\frac{d^Dk}{(2\pi)^D}\left\{[h_K(k)-h_\Delta(k)][g_K(k)+g_\Delta(k)]+\frac{d}{2}g_\Delta(k)h_\Delta(k)\right\}-\frac{d}{2}\int\frac{d^Dk}{(2\pi)^D}\left[\ln g_K(k)+\frac{g_\Delta(k)}{g_K(k)}\right]$$
$$-d\int_{q=k_1+k_2}\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\Delta_\mu(q)(\mathbf{k}_1\cdot\mathbf{k}_2)^2\{[g_K(k_1)+g_\Delta(k_2)][g_K(k_2)+g_\Delta(k_2)]-g_\Delta(k_1)g_\Delta(k_2)\}$$
$$+\frac{u}{(2\pi)^d}\int d^D\sigma\left[\frac{\pi}{K(\sigma)}\right]^{d/2} \qquad (2.15)$$

and the two-point correlation function $K(\sigma)$ reduces to

$$K(\sigma)=\int\frac{d^Dk}{(2\pi)^D}[1-\cos(\mathbf{k}\cdot\boldsymbol{\sigma})][g_K(k)+g_\Delta(k)]\ . \qquad (2.16)$$

For disorder fluctuations [8] we also introduce the function $L(\sigma)$,

$$L(\sigma)=\left\langle\frac{1}{2d}\langle (X^i(\sigma)-X^i(0))\rangle^2\right\rangle_{\rm dis}$$
$$=\int\frac{d^Dk}{(2\pi)^D}[1-\cos(\mathbf{k}\cdot\boldsymbol{\sigma})]g_\Delta(k)\ . \qquad (2.17)$$

Here $\langle\ \rangle$ means the thermal average. Note that the second equality is correct only in the Gaussian variational approximation. These two functions are characterized by two exponents ω and ω': $K(\sigma)\sim A_0\sigma^\omega$ and $L(\sigma)\sim A_0'\sigma^{\omega'}$.

Taking the variational derivatives of (2.15) with respect to $g_K(k)$ and $g_\Delta(k)$ and setting the results equal to zero, we find that

$$\frac{1}{g_K}=h_K-4\int\frac{d^Dq}{(2\pi)^D}\Delta_\mu(q)g_K(k-q)[(\mathbf{k}-\mathbf{q})\cdot\mathbf{k}]^2$$
$$-\frac{u}{2(2\pi)^d}\int d^D\sigma\left[\frac{1}{K(\sigma)}\right]^{d/2+1}(\pi)^{d/2}$$
$$\times[1-\cos(\mathbf{k}\cdot\boldsymbol{\sigma})] \qquad (2.18)$$

and

$$\frac{g_\Delta}{g_K^2}=h_\Delta+4\int\frac{d^Dq}{(2\pi)^D}\Delta_\mu(q)g_\Delta(k-q)[(\mathbf{k}-\mathbf{q})\cdot\mathbf{k}]^2\ . \qquad (2.19)$$

The saddle point equation (2.18) is different from that of the pure self-avoiding tethered membrane [20,21] by the second term. This term comes from the stress disorder term and, as we shall see below, we can also interpret it as the contribution from the fluctuations of the spin-glass operator in the short-range disorder case. We note that these saddle point equations have essentially the same structure as the integral equations studied in [8].

In the short-range disorder case, we can incorporate the spin-glass phase. As in [7], we consider the ground state $X_{a,\rm cl}^i$

$$\partial_\alpha X_{a,\rm cl}^i\partial_\beta X_{b,\rm cl}^j=q\delta_{\alpha\beta}\delta_{ij}\quad(a\neq b) \qquad (2.20)$$

and fluctuations about the ground state

$$X_a^i(\sigma)=X_{a,\rm cl}^i(\sigma)+\delta X_a^i(\sigma)\ . \qquad (2.21)$$

The thermal average is calculated by use of the trial Hamiltonian

$$\mathcal{H}_{\rm var}=\tfrac{1}{2}\int d^Dk(X_a^i(-k)-X_{a,\rm cl}^i(-k))$$
$$\times G_{ab}^{-1}(k)(X_b^i(k)-X_{b,\rm cl}^i(k))\ . \qquad (2.22)$$

Then, we have

$$\frac{1}{L^D}\left\langle-\tfrac{1}{2}\sum_{a,b=1}^{n}\Delta_\mu\int d^D\sigma\,\partial_\alpha X_a^i\partial_\alpha X_a^i\partial_\beta X_b^j\partial_\beta X_b^j\right\rangle_{\rm var}$$
$$=-2d\sum_{a\neq b=1}^{n}\Delta_\mu q\int\frac{d^Dk}{(2\pi)^D}k^2G_{ab}(k)-\sum_{a\neq b=1}^{n}d\Delta_\mu Dq^2-\tfrac{1}{2}d^2\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_3}{(2\pi)^D}[n^2\Delta_\mu\mathbf{k}_1^2\mathbf{k}_3^2G_{aa}(k_1)G_{bb}(k_3)]$$
$$-\tfrac{1}{2}d^2\int_{q=k_1+k_2}\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\Delta_\mu\sum_{a,b}[2(\mathbf{k}_1\cdot\mathbf{k}_2)^2G_{ab}(k_1)G_{ab}(k_2)]\ . \qquad (2.23)$$

In this case, the saddle point equations are

$$\frac{1}{g_K}=h_K+2\Delta_\mu q k^2-4\int\frac{d^Dq}{(2\pi)^D}\Delta_\mu g_K(k-q)[(\mathbf{k}-\mathbf{q})\cdot\mathbf{k}]^2-\frac{u}{2(2\pi)^d}\int d^D\sigma\left[\frac{1}{K(\sigma)}\right]^{d/2+1}(\pi)^{d/2}[1-\cos(\mathbf{k}\cdot\sigma)] \tag{2.24}$$

and

$$\frac{g_\Delta}{g_K^2}=h_\Delta+4\int\frac{d^Dq}{(2\pi)^D}\Delta_\mu g_\Delta(k-q)[(\mathbf{k}-\mathbf{q})\cdot\mathbf{k}]^2+2\Delta_\mu q k^2 . \tag{2.25}$$

From these calculations, we can identify the contributions from the stress disorder with the fluctuation of the spin-glass operator. It is interesting to note that, except for the contributions from the stress disorder, these saddle point equations (2.24) and (2.25) are the same as those of the randomly polymerized membrane with long-range interactions [23].

III. SOLUTIONS TO SADDLE POINT EQUATIONS

In this section, we restrict our interest to the two-dimensional membranes in d dimensions and examine the large distance (infrared) behaviors of $K(\sigma)$, $L(\sigma)$, $g_K(k)$, and $g_\Delta(k)$. We first study the crumpled phases of self-avoiding tethered membranes with long-range disorders and then the crumpled-glass phase ($q\neq 0$) in the case of short-range disorder.

A. Large distance behaviors of crumpled phases

To find the possible phases we analyze the saddle point equations (2.18) and (2.19) through convergence criteria. We assume the following forms of $g_K(k)$, $g_\Delta(k)$, $K(\sigma)$, and $L(\sigma)$ in the infrared limit:

$$g_K(k)^{-1}\sim k_0 k^{2+\alpha} ,$$

$$g_\Delta(k)^{-1}\sim k_0' k^{2+\alpha'} ,$$

$$K(\sigma)\sim A_0\sigma^{\omega} ,$$

$$L(\sigma)\sim A_0'\sigma^{\omega'} .$$

The exponent ω is related to the standard exponent ν for the radius of gyration ($R_G^2\sim L^{2\nu}$) by $\nu=\omega/2$. We then have

$$\int\frac{d^2k}{(2\pi)^2}g_K(k)[1-\cos(\mathbf{k}\cdot\sigma)]\sim\sigma^{\alpha} ,$$
$$\int\frac{d^2k}{(2\pi)^2}g_\Delta(k)[1-\cos(\mathbf{k}\cdot\sigma)]\sim\sigma^{\alpha'} . \tag{3.1}$$

If $\alpha>\alpha'$, the thermal fluctuations take over disorder fluctuations; $K(\sigma)\sim\sigma^{\alpha}$ and $\omega=\alpha$ hold. We name such a regime a temperature-dominated phase [8]. On the other hand, if $\alpha<\alpha'$, disorder fluctuations take over the thermal fluctuations and two-point correlation function is determined by the disorder fluctuations. That is, $K(\sigma)\sim L(\sigma)\sim\sigma^{\alpha'}$ and $\omega=\alpha'$. We call such a regime disorder-dominated phase. These suggest that $\omega=\max(\alpha,\alpha')$. We now analyze the crumpled phases of the membrane. We assume the following constraints on the values of the above exponents:

$$0<\omega ,\quad \omega'<2 ,\quad 0<\alpha ,\quad \alpha'<2 . \tag{3.2}$$

Inserting the dominant behaviors $g_K(k)^{-1}\sim k^{2+\alpha}$ and $g_\Delta(k)^{-1}\sim k^{2+\alpha'}$ in the integrands, we find that

$$\int\frac{d^2q}{(2\pi)^2}\Delta_\mu(q)g_K(\mathbf{k}-\mathbf{q})[(\mathbf{k}-\mathbf{q})\cdot\mathbf{k}]^2\sim k^{4-Z_\mu-\alpha} , \tag{3.3}$$

$$\int\frac{d^2q}{(2\pi)^2}\Delta_\mu(q)g_\Delta(\mathbf{k}-\mathbf{q})[(\mathbf{k}-\mathbf{q})\cdot\mathbf{k}]^2\sim k^{4-Z_\mu-\alpha'} . \tag{3.4}$$

From the dominant behavior $K(\sigma)\sim\sigma^{\omega}$, we also find that

$$h_K(k)-\frac{u}{2(2\pi)^d}\int d^2\sigma\left[\frac{1}{K(\sigma)}\right]^{1+d/2}(\pi)^{d/2}\times[1-\cos(\mathbf{k}\cdot\sigma)]\sim k^{2+\theta} \tag{3.5}$$

and

$$\theta=-4+\left[1+\frac{d}{2}\right]\omega\le 2 . \tag{3.6}$$

The condition $\theta\le 2$ comes from the following. If $\theta>2$, the coefficient of the k^4 term on the left-hand side of (3.5) becomes finite and the infrared behavior is $\sim k^4$ [20], which implies $\theta=2$. From these considerations we determine the infrared behaviors, that is, ω, α, and α' of the membranes in several cases.

1. *Pure case (no disorder)*

For the readers' convenience, we summarize the results of the Gaussian variational approximation for the two-dimensional self-avoiding tethered membranes in d dimensions [20,21]. In this case, the equalities $\theta=\alpha=\omega$ hold and from (3.6) we obtain $\omega=8/d$. This means that the crumpled phases exist only at $d>4$ and the membrane at $d\le 4$ is in the flat phase. In [20] more careful discussions have been made in order that the Gaussian variational approximation works for the polymer case. However, their improvement results in that the two-dimensional self-avoiding tethered membrane in four-dimensions is in the crumpled phase, which contradicts with the results of the numerical simulations [17,21]. Therefore, we shall not pursue such a direction.

2. *Stress disorder only*

In this case, $\Delta_K(q)=0$, which means the up-down symmetry of the membrane. Then $g_\Delta(k)=0$ and the equality $\omega=\alpha$ holds. We only need to consider Eq. (2.18). Introducing two positive constants c_1 and c_2, we can write Eq. (2.18) as

$$g_K(k)^{-1}=c_1 k^{2+\theta}-c_2 k^{4-Z_\mu-\alpha} . \tag{3.7}$$

If $4-Z_\mu-\alpha>2+\theta$, then the disorder term becomes irrelevant and the infrared behavior is determined by the self-avoiding term. That is, the self-avoidance completely determines the behavior. Then, we can use the result of the case of no disorder and we obtain $\omega=8/d$. We put this result in the above condition to obtain the constraint

$$2-Z_\mu>\frac{16}{d} . \tag{3.8}$$

Then we may ask what happens when the above constraint does not hold, that is, when $16/d\ge 2-Z_\mu$ and the stress disorder becomes relevant. At first sight, the second term in Eq. (3.7) determines the infrared behavior of the membrane. However, this is not true. One should note that the coefficient of the second term in Eq. (3.7) is negative, implying the cancellation between the first term and the second term. Physically speaking, the second term comes from the stress disorder and the fact that it becomes relevant means that the membrane tends to shrink. Then the exponent of the first term in Eq. (3.7) becomes smaller and the infrared behavior of the membrane is determined by the condition that the exponents of the first term and the second term become equal. From this condition, we obtain

$$\alpha=\frac{6-Z_\mu}{2+d/2} . \tag{3.9}$$

This suggests that even in the short-range disorder case ($Z_\mu=0$), ω at $d<8$ is altered from the value $8/d$ of the pure case to $6/(2+d/2)$. We see that the self-avoiding tethered membranes with short-range stress disorder at $d>2$ are in the crumpled phase. This conclusion seems to contradict the result that for $T>0$ (short-range) stress and spontaneous curvature disorders are irrelevant in the flat phase of the membrane. It is not so. Our conclusion means only that the self-avoiding tethered membrane with short-range stress disorder is in the crumpled phase. It does not exclude its transition to the flat phase. Therefore, if the strength of the disorder is weak enough and we increase the rigidity of the membrane, the phase transition to the flat phase may occur. In a previous work, we gave a conjecture that the self-avoiding tethered membranes with short-range stress disorder are in the flat phase or in the crumpled-glass phase and they are not in the crumpled phase. There we assumed that the stress disorder is irrelevant even in the finite d case. This, however, does not hold, as we have seen above. We note that in the long-range disorder case ($Z_\mu>0$), the membrane does not become flat even at $d=2$. It is a drawback of the variational method.

3. *Curvature disorder only*

From Eq. (2.19) we find that $\alpha'=Z_K+2\alpha-2$ and the equality $\alpha=\theta$ holds, that is,

$$\alpha=-4+\left[1+\frac{d}{2}\right]\omega , \quad \omega=\max(\alpha,\alpha') . \tag{3.10}$$

In the temperature-dominated phase ($\alpha>\alpha'$), we find that $\omega=\alpha=8/d$ and the condition $\alpha>\alpha'$ reduces to $8/d<2-Z_K$. In the disorder-dominated phase ($\alpha'>\alpha$), the equality $\omega=\alpha'$ holds and

$$\alpha=\frac{4-(1+d/2)(Z_K-2)}{1+d} . \tag{3.11}$$

For the short-range curvature disorder ($Z_K=0$), at $d>4$, $\omega=8/d$ and at $d<4$, $\omega>\alpha=(6+d)/(1+d)>2$. This means that the crumpled phases do not exist at $d<4$ and the self-avoiding tethered membrane with short-range curvature disorder at $d<4$ is in the flat phase. However, long-range curvature disorder can destroy the flat phase. For example, in the case $Z_K=1$, $\omega=\alpha'=9/(1+d)$ at $d<8$. From this we see that the self-avoiding tethered membrane with long-range curvature disorder ($Z_K=1$) at $d=4$ can be in the crumpled phase.

4. *General case*

At first, we determine the value of the exponent α'. Introducing two positive constants d_1 and d_2, we can write Eq. (2.19) as

$$k^{2+2\alpha-\alpha'}=d_1 k^{4-Z_K}+d_2 k^{4-Z_\mu-\alpha'} . \tag{3.12}$$

Then, if $Z_K>Z_\mu+\alpha'$, the infrared behavior is determined by the first term of this equation and $\alpha'=Z_K+2\alpha-2$. Inserting this value in the above condition, we obtain the constraint

$$0>Z_\mu+2\alpha-2 . \tag{3.13}$$

If $Z_K<Z_\mu+\alpha'$, the second term determines the infrared behavior. Since

$$k^{2+2\alpha-\alpha'}\sim k^{4-Z_\mu-\alpha'} , \tag{3.14}$$

we find that $2-Z_\mu-2\alpha=0$. That is, the above constraint is not broken and the equality $\alpha'=Z_K+2\alpha-2$ holds always. However, as we shall see below, the constraint (3.13) is not necessarily conserved.

The process of determining the exponent α is essentially the same as in the previous cases. The different phases are characterized by whether the second term in Eq. (3.7) (the contribution of the stress disorder) is relevant or not and whether the phase is in the temperature-dominated phase or in the disorder-dominated phase; four cases are *a priori* possible.

(i) The first is the temperature-dominated ($\alpha>\alpha'$) and irrelevant stress disorder phase $\alpha=8/d$, $\alpha'=Z_K+16/d-2$. Then $K(\sigma)\sim\sigma^{\alpha}$. This phase exists for $16/d<2-Z_\mu$ and $Z_K<2-8/d$.

(ii) The next is the temperature-dominated ($\alpha>\alpha'$) and relevant stress disorder phase $\alpha=(6-Z_\mu)/(2+d/2)$, $\alpha'=Z_K+(8-d-2Z_\mu)/(2+d/2)$. Then $K(\sigma)\sim\sigma^{\alpha}$. This phase exists for $16/d>2-Z_\mu$ and $Z_K<(d-2+Z_\mu)/(2+d/2)$.

(iii) Then there is the disorder-dominated ($\alpha<\alpha'$) and irrelevant stress disorder phase $\alpha=[6+d-(1+d/2)Z_K]/(1+d)$, $\alpha'=(10-Z_K)/(1+d)$. Then $K(\sigma)\sim\sigma^{\alpha'}$. This phase exists for $Z_K>10/(2+d)+(1$

$+d)Z_\mu/(2+d)$, $Z_K>2-8/d$, and $8-Z_K<2d$.

(iv) Finally, there is the disorder-dominated ($\alpha<\alpha'$) and relevant stress disorder phase $\alpha=[8+d-Z_\mu-(1+d/2)Z_K]/(3+d)$, $\alpha'=(Z_K+10-2Z_\mu)/(3+d)$. Then $K(\sigma)\sim\sigma^{\alpha'}$. This phase exists for $Z_K<10/(2+d)+(1+d)Z_\mu/(2+d)$, $Z_K>(d-2+Z_\mu)/(2+d/2)$, and $Z_K+4-2Z_\mu<2d$.

In Figs. 1 and 2 we summarize the phase diagrams of the restricted cases of the general result. Figure 1 shows the phase diagram of the crumpled phases in (d,Z_K) plane for the case of the short-range stress disorder ($Z_\mu=0$). The region where the above crumpled phases do not occupy is considered to be the flat phase or the crumpled-glass phase. Figure 2 shows the phase diagram of the crumpled phases in (Z_μ,Z_K) plane for the case when the embedding dimension d is three. In Ref. [8] the phase diagram of the flat phases and flat glass phases of tethered membrane in (Z_μ,Z_K) plane for $d=3$ case is given. Our conclusion is that the tethered membrane with disorder exponents (Z_μ,Z_K) with no rigidity is in the associated crumpled phase of Fig. 2. Following the first scenario presented in the Introduction, when we increase the rigidity of the membrane the phase transition to the flat phases presented in the Ref. [8] or to the crumpled glass phase occurs, depending on whether the strength is weak or strong. If we follow the second scenario [8], the exponents (Z_μ,Z_K) must be larger than the critical values in order that the phase transition to the crumpled-glass phase occurs.

B. Large distance behaviors of the crumpled-glass phases (short-range disorders)

In the crumpled-glass phase ($q\neq0$), we need to study the saddle point equations (2.24) and (2.25). The analysis proceeds in the same way as in the preceding subsection. From Eq. (2.25) we find that the infrared behavior of $g_\Delta(k)$ is completely determined by the last term $2\Delta_\mu qk^2$. Then, we have $\alpha'=2\alpha$, indicating that the membrane is in the disorder-dominated phase ($\alpha>\alpha'$). This is very natural because the membrane is in the crumpled-glass phase and the infrared behavior is determined by the disorder of the ground state. With the above result ($\omega=\alpha'=2\alpha$), the analysis can be done as in the stress disorder only case. We present the results. At $d>3$, the stress disorder is irrelevant and $\omega=8/(1+d)$. At $d\leq3$, both the self-avoidance and the disorder are relevant. However, ω becomes larger than 2 and we believe that the crumpled-glass phase does not exist at $d\leq3$. These situations entirely coincide with the previous ones [23] by changing γ with d. The fluctuations of the spin-glass operator does not modify the previous result in contrast to the crumpled phases.

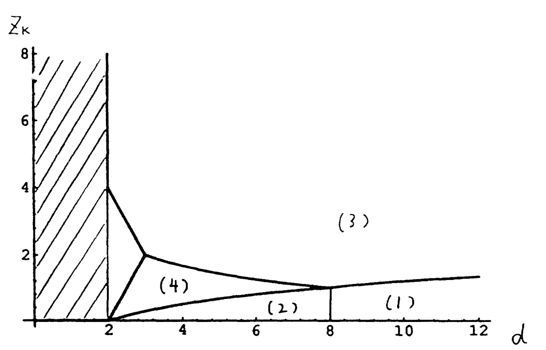

FIG. 1. The crumpled phases of the self-avoiding tethered membrane with short-range stress disorder ($Z_\mu=0$) and long-range curvature disorder with exponent Z_K in d-dimensional space: (1) temperature-dominated and irrelevant stress disorder phase; (2) temperature-dominated and relevant stress disorder phase; (3) disorder-dominated and irrelevant stress disorder phase; and (4) disorder-dominated and relevant stress disorder phase.

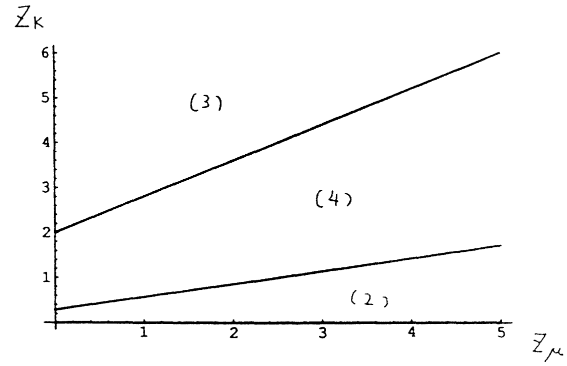

FIG. 2. The crumpled phases of the self-avoiding tethered membrane with long-range stress disorder with exponent Z_μ and long-range curvature disorder with exponent Z_K in three-dimensional space: (2) temperature-dominated and relevant stress disorder phase; (3) disorder-dominated and irrelevant stress disorder phase; and (4) disorder-dominated and relevant stress disorder phase.

IV. DISCUSSIONS

In this paper we have studied the crumpled phases of the self-avoiding tethered membrane with disorders. We have shown that even a short-range stress disorder is relevant at $d<8$ and the crumpled phase exists even at $2<d\leq4$. This result is in contrast with the crumpled phase of the phantom tethered membrane [5]. There, a short-range stress disorder merely causes the membrane to swell slightly and should not affect the universal property. We have given the exponent for the radius of gyration. Furthermore, we have considered the case where the disorders have long-range correlations. The membrane's behavior is characterized by whether the stress disorder is relevant or not and whether the membrane is in the temperature-dominated phase or in the disorder-dominated phase. Specifying which phases the membrane with the disorder exponents (Z_μ,Z_K) in d dimensions belongs to, we have given the exponent for the radius of gyration. The phase diagrams in two special cases are presented (Figs. 1 and 2). We have also studied the behavior of the crumpled-glass phase in the case of short-range disorders. In this case, the contribution from stress disorder can be seen as the fluctuations of the spin-glass operator. This fluctuation does not modify the

behaviors of the crumpled-glass phase. However, the analysis is restricted to the replica symmetric solution and the improvement is left for a future study.

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APPENDIX: THE GAUSSIAN APPROXIMATION FOR SELF-AVOIDING TETHERED MEMBRANES

We shall discuss the Gaussian variational approximation [20,21,27] for the self-avoiding tethered membrane in the flat phase. The self-avoidance is always considered to be irrelevant when the membrane is in the flat phase. This picture seems correct intuitively. However, if the self-avoidance is irrelevant when the membrane is flat, what makes the membrane flat? In order to answer this question, we apply the Gaussian variational method to the flat phase of the self-avoiding tethered membrane and show the complete equivalence with the tethered membrane with long-range interaction. When the membrane is crumpled, it is well known [20,21] that the tethered membranes with long-range force $1/r^d$ in the large embedding space dimension limit are equivalent to the self-avoiding membranes in d dimensions in the variational approximation. We show that this correspondence remains to be correct even in the flat phase of the membrane, namely, the self-avoidance is relevant even in the flat phase of the membranes.

We consider the generalized Edwards model

$$\mathcal{H}[\mathbf{X}(\sigma)]=\int d^D\sigma \tfrac{1}{2}\partial_\alpha X^i \partial_\alpha X^i + u\int d^D\sigma \int d^D\sigma' \delta^d(X^i(\sigma)-X^i(\sigma')) \,. \quad (A1)$$

The best quadratic Hamiltonian is determined by finding an upper bound for the exact free energy $\mathcal{F}$,

$$\mathcal{F}=-\ln Z \;,\quad Z=\int \mathcal{D}[\mathbf{X}]e^{-\mathcal{H}} \,. \quad (A2)$$

The possibility of symmetry breaking suggests that $\mathbf{X}$ can be decomposed into transverse (Goldstone) mode $\mathbf{h}(\sigma)$ and phonon modes $u^i(\sigma)$ [20],

$$\mathbf{X}(\sigma)=[\zeta\sigma^i+u^i(\sigma)]\mathbf{e}_i+\mathbf{h}(\sigma) \;,\quad \mathbf{h}\cdot\mathbf{e}_i=0 \,. \quad (A3)$$

Thus the most general trial Hamiltonian, which is quadratic in the fields, is

$$\mathcal{H}_0=\int d^Dk\{u^i(-k)g_1(k)u^i(k)+\mathbf{h}(-k)g_2(k)\cdot\mathbf{h}(k)\} \,. \quad (A4)$$

As has been discussed in [21], the original Hamiltonian is invariant under rotations in the embedding space. Therefore, for $\zeta>0$, we must integrate over all possible ground states in addition to the integrations over $u^i(k)$ and $\mathbf{h}(k)$. In other words, the average should be taken over all possible orientations of the flat phase and the average value $\langle \mathbf{X}(\sigma)\rangle$ will vanish. This is the most important point when we carry out the Gaussian variational approximation.

The variational parameters are the kernels $g_1(k), g_2(k)$ and the flatness order parameter ζ. The variational free energy $\mathcal{F}_v$, which is an upper bound for $\mathcal{F}$, is then given by

$$\mathcal{F}_v=\mathcal{F}_0+\langle \mathcal{H}-\mathcal{H}_0\rangle_0 \,. \quad (A5)$$

Here we define the average $\langle A \rangle_0$ by

$$\langle A \rangle_0 \equiv \int \mathcal{D}[\mathbf{X}]e^{-\mathcal{H}_0}A/Z_0, \qquad Z_0=\int \mathcal{D}[\mathbf{X}]e^{-\mathcal{H}_0} \,, \quad (A6)$$

and $\mathcal{F}_0$ by

$$\mathcal{F}_0=-\ln Z_0 \,. \quad (A7)$$

The calculation of $\mathcal{F}_v$ is easy except the evaluation of a term $\langle u\int d^D\sigma\int d^D\sigma'\delta^d(X^i(\sigma)-X^i(\sigma'))\rangle_0$. We describe it in some details. If we naively calculate the integration with the decomposition of X^i in (A3), we obtain

$$\left\langle u\int d^D\sigma\int d^D\sigma'\delta^d(X^i(\sigma)-X^i(\sigma'))\right\rangle_0 = uL_D\int d^D\sigma\int\frac{d^dk}{(2\pi)^d}\langle e^{i\mathbf{k}\cdot(\mathbf{X}(\sigma)-\mathbf{X}(0))}\rangle_0$$
$$= uL^D\int d^D\sigma\int\frac{d^dk}{(2\pi)^d}\prod_{i=1}^{D}\prod_{j=D+1}^{d}\langle e^{ik^i[u^i(\sigma)+\zeta\sigma^i-u^i(0)]+ik^j[h^j(\sigma)-h^j(0)]}\rangle_0 \,. \quad (A8)$$

For convenience, we introduce the following quantities:

$$K_1(\sigma)=\frac{1}{2D}\sum_{i=1}^{D}\langle (u^i(\sigma)-u^i(0))^2\rangle_0 = \int\frac{d^Dk}{(2\pi)^D}[1-\cos(\mathbf{k}\cdot\sigma)]\frac{1}{g_1(k)} \,, \quad (A9)$$

$$K_2(\sigma)=\frac{}{2(d-D)}\sum_{j=D+1}^{d}\langle (h^j(\sigma)-h^j(0))^2\rangle_0 = \int\frac{d^Dk}{(2\pi)^D}[1-\cos(\mathbf{k}\cdot\sigma)]\frac{1}{g_2(k)} \,. \quad (A10)$$

Then we find that Eq. (A8) is equal to

$$uL^D\int\frac{d^D\sigma}{(2\pi)^d}e^{-\zeta^2\sigma^2/4K_1(\sigma)}\left[\frac{\pi}{K_1(\sigma)}\right]^{D/2}\times\left[\frac{\pi}{K_2(\sigma)}\right]^{(d-D)/2} \,. \quad (A11)$$

From this equation we see in the phase with broken rotational symmetry ($\zeta\neq 0$), the self-avoidance term becomes completely irrelevant. However, as discussed before, this calculation is not correct because the above calculation destroys the rotational symmetry of the original Hamiltonian. In order to obtain the correct expression for this thermal average, we have to average over the all possible ground states. We return to the first expression of (A8) and take the average over the ground states. The average over the ground states means that we integrate the vector $\mathbf{X}(\sigma)-\mathbf{X}(0)$ over a sphere with radius $\sqrt{[\mathbf{X}(\sigma)-\mathbf{X}(0)]^2}$. More precisely, one ground state corresponds to a point on the Grassmann manifold $\mathrm{Gr}(D,d)$, and the above averaging procedure is not necessarily correct. However, this difference only causes an overall factor which is not essential. Therefore we replace the average over the ground states with the average over the sphere. Then

$$\left[uL^D\int d^D\sigma\int\frac{d^dk}{(2\pi)^d}\langle e^{i\mathbf{k}\cdot(\mathbf{X}(\sigma)-\mathbf{X}(0))}\rangle_0\right]_{\mathrm{g.s}}=uL^D\int d^D\sigma\int\frac{d^dk}{(2\pi)^d}\langle[e^{i\mathbf{k}\cdot(\mathbf{X}(\sigma)-\mathbf{X}(0))}]_{\mathrm{g.s.}}\rangle_0\ . \tag{A12}$$

The average over the ground states $[\ ]_{\mathrm{g.s.}}$ is calculated as

$$\begin{aligned}[e^{i\mathbf{k}\cdot(\mathbf{X}(\sigma)-\mathbf{X}(0))}]_{\mathrm{g.s.}}&=\int_{S^{d-1}}d\mathbf{n}\,e^{i\mathbf{k}\cdot\mathbf{n}\sqrt{(\mathbf{X}(\sigma)-\mathbf{X}(0))^2}}\\&=\int_{S^{d-1}}d\mathbf{n}\,e^{i\mathbf{k}\cdot\mathbf{n}\sqrt{(\mathbf{X}(\sigma)-\mathbf{X}(0))^2}}\\&=\int d\mathbf{n}\int_{-i\infty}^{+i\infty}d\lambda\,e^{\lambda(\mathbf{n}^2-1)}e^{i\mathbf{k}\cdot\mathbf{n}\sqrt{(\mathbf{X}(\sigma)-\mathbf{X}(0))^2}}\\&=\mathrm{const}\times\int d\lambda\,e^{-\mathbf{k}^2\times(\mathbf{X}(\sigma)-\mathbf{X}(0))^2/4\lambda}e^{-\lambda}\ .\end{aligned}\tag{A13}$$

After rescaling the wave vector $\mathbf{k}$, we obtain that Eq. (A12) is equal to

$$L^d\times\mathrm{const}\times\frac{u}{(2\pi)^d}\int d^D\sigma\int\frac{d^dk}{(2\pi)^d}\langle e^{-\mathbf{k}^2\times(\mathbf{X}(\sigma)-\mathbf{X}(0))^2/2}\rangle_0=\mathrm{const}\times\frac{u}{(2\pi)^d}\times L^D\int d^D\sigma\left[\frac{\pi}{K(\sigma)}\right]^{d/2}, \tag{A14}$$

where $K(\sigma)$ is defined by

$$K(\sigma)=\tfrac{1}{2}\langle(\mathbf{X}(\sigma)-\mathbf{X}(0))^2\rangle_0=\tfrac{1}{2}\zeta^2\sigma^2+DK_1(\sigma)+(d-D)K_2(\sigma)\ . \tag{A15}$$

When we calculate the thermal average $\langle\ \rangle_0$ of the other terms in the variational free energy, we have to carry out the average over the ground states in the same way. But this causes only an overall factor which is not relevant in the variational analysis and can be dropped. Then the variational free energy is evaluated as

$$\begin{aligned}\mathcal{F}_v&=\mathcal{F}_0+\langle\mathcal{H}-\mathcal{H}_0\rangle\\&=L^D\left[\frac{1}{2}D\zeta^2+\int\frac{d^dk}{(2\pi)^d}\left[\frac{k^2}{2g_1(k)}-1\right]+(d-D)\int\frac{d^dk}{(2\pi)^d}\left[\frac{k^2}{2g_2(k)}-1\right]+\frac{u'}{(2\pi)^d}\int d^D\sigma\left[\frac{\pi}{K(\sigma)}\right]^{d/2}\right.\\&\quad\left.+D\int\frac{d^dk}{(2\pi)^d}\ln g_1(k)+(d-D)\int\frac{d^dk}{(2\pi)^d}\ln g_2(k)\right]\ .\end{aligned}\tag{A16}$$

Taking the variational derivatives of the above free energy with respect to $g_1(k)$, $g_2(k)$, and ζ and setting the results equal zero, we arrive at

$$\begin{aligned}g_1(k)&=\frac{1}{2}k^2-\frac{d}{2}\frac{u'}{(2\pi)^d}\int d^D\sigma\left[\frac{1}{K(\sigma)}\right]^{d/2+1}(\pi)^{d/2}[1-\cos(\mathbf{k}\cdot\boldsymbol{\sigma})]\ ,\\g_2(k)&=\frac{1}{2}k^2-\frac{d}{2}\frac{u'}{(2\pi)^d}\int d^D\sigma\left[\frac{1}{K(\sigma)}\right]^{d/2+1}(\pi)^{d/2}[1-\cos(\mathbf{k}\cdot\boldsymbol{\sigma})]\ ,\\\frac{D}{2}&=\frac{d}{4}u'\int d^D\sigma\left[\frac{\pi}{K(\sigma)}\right]^{d/2+1}\sigma^2\ .\end{aligned}\tag{A17}$$

From the last equation in (A17), we see that in the flat phase ($\zeta>0$), k^2 coefficients of $g_1(k)$ and $g_2(k)$ vanish. If we assume that two propagators g_1 and g_2 are equal, we obtain essentially the same equation as in the work by Guitter and Palmeri [20]. That is, the self-avoiding tethered membrane in d dimensions in the variational Gaussian approximation is the same as the tethered membrane with long-range repulsive interaction (varying as $1/r^d$) in the large embedding space dimension limit. We note that, considering the rotational symmetry of the Hamiltonian, we have taken the average over the ground states. This process is necessary because the phase space should also have the rotational symmetry. However, this point may be controversial. A more convincing way of deriving the variational free energy (A16) is the Legendre transformation, but this procedure is very involved.

[1] M. Mutz, D. Bensimon, and M. J. Brienne, Phys. Rev. Lett. **67**, 923 (1991).
[2] D. Bensimon, D. Mukamel, and L. Peliti, Europhys. Lett. **18**, 269 (1992).
[3] D. Bensimon, M. Mutz, and T. Gulik, Physica A **194**, 190 (1993).
[4] D. R. Nelson and L. Radzihovsky, Europhys. Lett. **16**, 79 (1991).
[5] L. Radzihovsky and D. R. Nelson, Phys. Rev. A **44**, 3525 (1991).
[6] D. C. Morse, T. C. Lubensky, and G. S. Grest, Phys. Rev. **45**, 2151 (1992).
[7] L. Radzihovsky and P. Le. Doussal, J. Phys. I **2**, 599 (1992).
[8] P. Le. Doussal and L. Radzihovsky, Phys. Rev. B **48**, 3548 (1993).
[9] Y. Kantor, Europhys. Lett. **20**, 337 (1992).
[10] M. Plischke and D. Boal, Phys. Rev. A **38**, 4943 (1988).
[11] F. F. Abraham, W. Rudge, and M. Plischke, Phys. Rev. Lett. **62**, 1757 (1989).
[12] J.-S. Ho and A. Baumgarter, Phys. Rev. Lett. **63**, 1234 (1989).
[13] D. Boal, E. Levinson, D. Liu, and M Plischke, Phys. Rev. A **40**, 3292 (1989).
[14] A. Baumgarter and J.-S. Ho, Phys. Rev. A **41**, 5747 (1990).
[15] J.-S. Ho and A. Baumgarter, Europhys. Lett. **12**, 295 (1990).
[16] F. F. Abraham and D. R. Nelson, J. Phys. (Paris) **51**, 2653 (1990).
[17] G. S. Grest, J. Phys. I **1**, 1695 (1991).
[18] A. Baumgarter and W. Renz, Europhys. Lett. **17**, 381 (1992). In this work, a different model (plaquette membrane instead of hard sphere membrane) for the self-avoiding tethered membrane is used and they concluded that the membrane is crumpled.
[19] D. M. Kroll and G. Gompper, J. Phys. I **3**, 1131 (1993). In this work, self-avoiding plaquette membranes are shown to be in the flat phase.
[20] E. Guitter and J. Palmeri, Phys. Rev. A **45**, 734 (1992).
[21] M. Goulian, J. Phys. II **1**, 1327 (1991).
[22] P. Le. Doussal, J. Phys. A **25**, L469 (1992).
[23] S. Mori and M. Wadati, Phys. Lett. A **185**, 206 (1994).
[24] *Statistical Mechanics of Membranes and Surfaces,* Proceedings of the Fifth Jerusalem Winter School, edited by D. R. Nelson, T. Piran, and S. Weinberg (World Scientific, Singapore, 1989).
[25] S. F. Edwards and P. W. Anderson, J. Phys. (Paris) F **5**, 965 (1975).
[26] M. Mezard and G. Parisi, J. Phys. I **1**, 809 (1991).
[27] J. des Cloizeaux and G. Jannik, *Polymers in Solution* (Clarendon, Oxford, 1990).